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Enforcing Embedded Interface and Boundary Conditions by Nitsche's Method

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The specification of Dirichlet boundary conditions or interface constraints often relies on mesh alignment with geometric features, as well as interpolatory properties of finite element functions. Weak enforcement of constraints relaxes these requirements, allowing the consideration of convenient alternative approaches. Classical techniques for weak enforcement of constraints are Lagrange multiplier and penalty methods. Lagrange multiplier techniques involve auxiliary fields, incurring additional computational cost and susceptibility to severe stability restrictions. Penalty methods are often derived in ad hoc manner, with attendant sensitivity to numerical parameters and potential ill conditioning with mesh refinement. The Nitsche method was conceived originally as a penalty method, modified for variational consistency [1]. It is constructive to view this alternative approach as emanating from a stabilized Lagrange multiplier formulation, in which flux-like terms replace the Lagrange multipliers [2]. This procedure eliminates the auxiliary field, converting the classical penalty parameter into a stabilization parameter, with enhanced robustness.

A key challenge in employing non-interpolatory basis functions for finite element methods is the robust imposition of Dirichlet boundary conditions. Smooth functions such as rational B-splines are becoming increasingly popular for use in finite element approximations to solutions of partial differential equations. This is due in part to their ability to provide exact geometric representations of curved domains, but also to an increasing interest in fourth-order problems. Since B-splines are not interpolants, there is no overriding need for the mesh to conform to the boundary. Nitsche's method provides an attractive mechanism for enforcing Dirichlet boundary conditions for B-spline bases, working well both when the mesh conforms to the boundary and in the more general case when it does not. In the latter case, partial elements (knot-spans associated with the splines) are addressed by modified quadrature schemes. The stabilization parameter is designed to provide coercivity of the discrete bilinear form, computed by solving local eigenvalue problems associated with each element that intersects or abuts on the Dirichlet boundary. For non-conforming meshes the eigenvalue problem is modified to bound the eigenvalues.

Nitsche's method, devised originally for second-order problems, is extended to handle B-spline bases for fourth-order problems, in which Dirichlet boundary conditions may involve both the primary function and its derivative. In this case the method uses a stabilization parameter for each type of Dirichlet boundary condition. Here too, the stabilization parameters are obtained by solving element-level eigenvalue problems along the Dirichlet boundary, so as to provide coercivity of the discrete bilinear form, and similar mechanisms are employed to handle non-conforming meshes. The accuracy of the approach is confirmed numerically for representative second- and fourth-order problems, exhibiting optimal rates of convergence in all cases. A set of parametric studies investigates the influence of numerical values of the stabilization parameters on the quality of the computed results. A range of 10 orders of magnitude yields minimal errors for the second-order problems examined, and two orders of magnitude for the fourth-order problem considered. In all cases, the local eigenvalue procedure advocated to define the stabilization parameters provides values well within the optimal ranges. Another parametric study demonstrates the robustness of the approach with respect to the location of non-conforming grids relative to the domain.

On a related topic, a stabilized finite element method based on the Nitsche technique for enforcing constraints leads to an efficient computational procedure for embedded interface problems, in which the finite element mesh need not be aligned with the interface geometry. Interface problems in which the surface geometry is complex or evolving pose a substantial challenge to computation. Approaches with moving meshes have made considerable progress, e.g. [3], yet methods in which the interface is embedded in a fixed mesh offer an efficient alternative. Nitsche's method has been used to weakly enforce interface constraints along with partitioning of elements in the vicinity of the interface [4], akin to X-FEM enrichment schemes. In the present work two fundamental types of interface problems are considered. One is the case of "jump" problems where the discontinuity in both the primary field and normal flux across the interface are prescribed. The second is the case of "Dirichlet" problems in which the primary field itself is prescribed on both sides of the interface, and the unknown jump in interfacial flux is a consequential quantity of interest. Procedures for accurate flux evaluations are provided. For the linear basis functions considered here, the stabilization parameters are defined explicitly by simple, parameter-free, element-level algebraic expressions, leading to an efficient finite element method that is robust for a broad class of interface problems. Stability analysis and a-priori error estimates are provided. Numerical examples (e.g. Fig.1) demonstrate the effectiveness of the proposed methodology, exhibiting optimal rates of convergence.

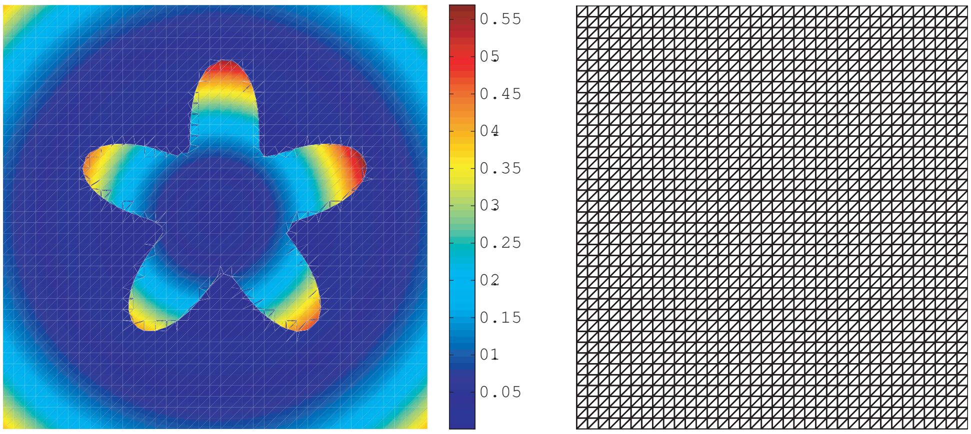

Figure 1: Finite-element solution (left) of a curved interface problem over a structured triangular mesh.

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